Test 3, NM2

Exc 1 $(f,g) = \int_{-1}^{1} |x| f(x) g(x) dx$

a. • $\varphi_0(x) = 1$

• $\varphi_1(x) = x - \alpha \varphi_0(x) = x - \alpha$ $\quad (\varphi_0, \varphi_1) = 0$ orthogonal

$(\varphi_1, \varphi_0) = (x-\alpha, 1) = (x,1) - (\alpha, 1)$

$= \underbrace{\int_{-1}^{1} |x| x \, dx}_{=0} - \alpha \underbrace{\int_{-1}^{1} |x| dx}_{\neq 0} = 0 \Rightarrow \alpha = 0$

due to symmetry interval and integrand even

$\Rightarrow \varphi_1(x) = x$

• $\varphi_2(x) = x^2 - \alpha \varphi_1(x) - \beta \varphi_0(x)$ $\quad \left.\begin{array}{l}(\varphi_2, \varphi_1) = 0 \\ (\varphi_2, \varphi_0) = 0\end{array}\right\}$ orthogonal

$(\varphi_2, \varphi_1) = (x^2, \varphi_1) - \alpha(\varphi_1, \varphi_1) - \beta \underbrace{(\varphi_0, \varphi_1)}_{\text{orthogonal},\ =0}$

$= \underbrace{\int_{-1}^{1} |x| x^3 dx}_{=0} - \alpha \underbrace{\int_{-1}^{1} |x| x^2 dx}_{\neq 0} = 0 \Rightarrow \alpha = 0$

symmetric interval and integrand even

$\Rightarrow \varphi_2(x) = x^2 - \beta \varphi_0(x) = x^2 - \beta$

$(\varphi_2, \varphi_0) = (x^2, \varphi_0) - (\beta, \varphi_0) = \int_{-1}^{1} |x| x^2 dx - \beta \int_{-1}^{1} |x| dx = 0$

$\Rightarrow 2 \int_0^1 x^3 dx - 2\beta \int_0^1 x \, dx = 0 \Rightarrow 2\left(\frac{1}{4} - \frac{1}{2}\beta\right) = 0 \Rightarrow \beta = \frac{1}{2}$

$\Rightarrow \varphi_2(x) = x^2 - \frac{1}{2}$

b. • Gauss rule can be used for $\int_{-1}^{1} |x| f(x) dx$

• Quadratic polynomial $\varphi_2(x) = x^2 - \frac{1}{2} \Rightarrow$ zeros $x_0 = -\frac{1}{\sqrt{2}}$, $x_1 = \frac{1}{\sqrt{2}}$

Langrange interpolating polynomial ; $l_0(x) = \frac{x - x_1}{x_0 - x_1}$, $l_1(x) = \frac{x - x_0}{x_1 - x_0}$

through $x_0, x_1 \Rightarrow$ first order polynomial

$P_1(x) = f(-\frac{1}{\sqrt{2}}) l_0(x) + f(\frac{1}{\sqrt{2}}) l_1(x)$ $\quad l_0(x) = \frac{x - \frac{1}{\sqrt{2}}}{-\sqrt{2}}$, $l_1(x) = \frac{x + \frac{1}{\sqrt{2}}}{\sqrt{2}}$

needs exact integration

$\int_{-1}^{1} |x| P_1(x) dx = f(-\frac{1}{\sqrt{2}}) \int_{-1}^{1} |x| l_0(x) dx + f(\frac{1}{\sqrt{2}}) \int_{-1}^{1} |x| l_1(x) dx$

hence:

Gauss rule: $w_0 f(-\frac{1}{\sqrt{2}}) + w_1 f(+\frac{1}{\sqrt{2}})$, $w_i = \int_{-1}^{1} |x| l_i(x) dx \quad i = 0,1$

$$\int_{-1}^{1} |x|\, l_0(x)\, dx = -\frac{1}{\sqrt{2}} \int_{-1}^{1} |x| \left(x - \frac{1}{\sqrt{2}}\right) dx = -\frac{1}{\sqrt{2}} \Big(\underbrace{\int_{-1}^{1} |x| x\, dx}_{=0 \text{ symmetry}} - \frac{1}{\sqrt{2}} \underbrace{\int_{-1}^{1} |x|\, dx}_{=2\int_0^1 x\, dx = 1} \Big)$$

$$= -\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\int_{-1}^{1} |x|\, l_1(x)\, dx = \frac{1}{\sqrt{2}} \int_{-1}^{1} |x| \left(x + \frac{1}{\sqrt{2}}\right) dx = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$\Rightarrow$ Gauss rule : $\frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(+\frac{1}{\sqrt{2}}\right) = \int_{-1}^{1} |x|\, f(x)\, g(x)$

c. Gauss rule exact:

if max. degree of orthogonal polynomials used is n
than Gauss rule degree of exactness: $2n-1$

now $n=2$ (second-degree polynomial) $\Rightarrow$ $2.2-1=3$
n nodes
degree of exactness

Check:

$f(x)=1$
- $\int_{-1}^{1} |x| f(x)\, dx = \int_{-1}^{1} |x|\, dx = 1$
- $\frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}.1 + \frac{1}{2}.1 = 1$

} are similar

$f(x)=x$
- $\int_{-1}^{1} |x|\, x\, dx = 0$
- $\frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \cdot \frac{-1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = 0$

} are similar

$f(x)=x^2$
- $\int_{-1}^{1} |x|\, x^2\, dx = 2\int_0^1 x^3\, dx = \frac{1}{2}$
- $\frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

} are similar

$f(x)=x^3$
- $\int_{-1}^{1} |x|\, x^3\, dx = 0$
- $\frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^3 = 0$

} are Similar

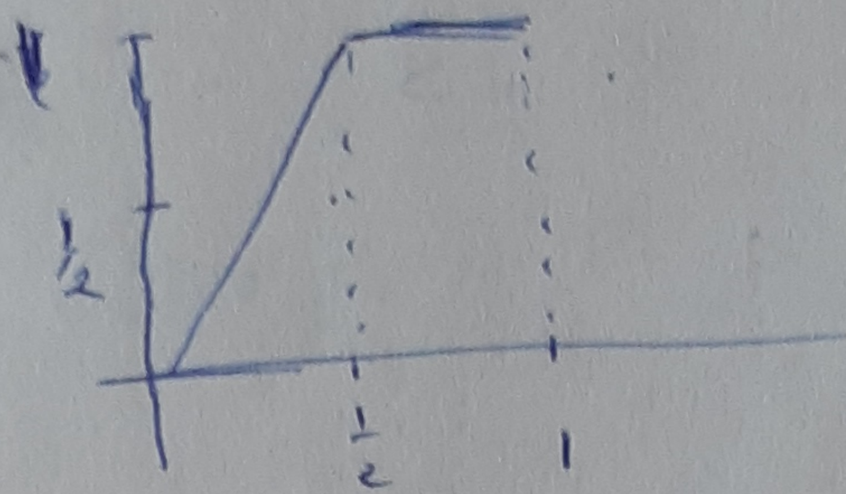
$f(x)=x^4$
- $\int_{-1}^{1} |x|\, x^4\, dx = 2\int_0^1 x^5\, dx = \frac{1}{3}$
- $\frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2} f\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)^4 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4}$

} are not similar

$\Rightarrow$ degree of exactness is 3
$\Rightarrow$ agrees with the theory

Exc 2 $\quad f(x)$ on $[0,1]$ $\quad f(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}$

(a) sketch of f



best linear approximation $p_1(x)$

Chebyshev equioscillation theorem

best approximation p_n^* of f exists and unique and $\exists$ n+2 points $x_0 < x_1 < \dots < x_{n+1}$ s.t.

$$f(x_j) - p_n^*(x_j) = \sigma(-1)^j E_n^*(f) \quad j = 0, \dots, n+1$$

with $\sigma = 1$ or $\sigma = -1$ (depending on f, n)

and $E_n^*(f) = \| f - p_n^* \|_\infty$

now $n = 1$, error has equal extrema according to theorem

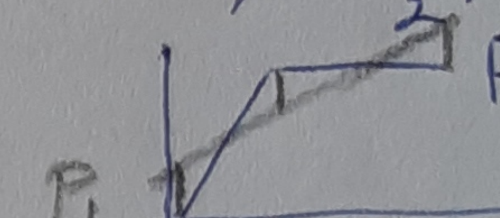
$\Rightarrow \quad p_1(x) = ax + b$

$\Rightarrow \quad p_1(0) - f(0) = b$

$p_1(\frac{1}{2}) - f(\frac{1}{2}) = \frac{1}{2}a + b - 1$

$p_1(1) - f(1) = a + b - 1$

extrema error in $x = 0$, $x = \frac{1}{2}$, $x = 1$

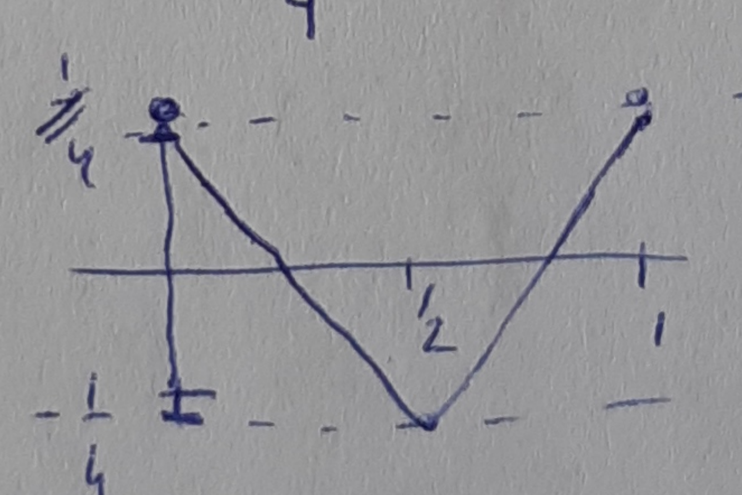


$\Rightarrow \quad b = -(\frac{1}{2}a + b - 1) \quad \Rightarrow \quad 2b = -\frac{1}{2}a + 1 \quad \Rightarrow \quad b = \frac{1}{4}$

$b = a + b - 1 \quad \Rightarrow \quad a = 1$

$\Rightarrow \quad p_1(x) = x + \frac{1}{4}$

error plot:



(b) Chebyshev approximation on $[-1, 1]$ is

$C_n(x) = \sum_{k=0}^{n} a_k T_k(x)$ with $a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) T_k(x)\, dx$

$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx$

$(f, g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) g(x)\, dx$

$T_k(x) = \cos k\theta$

$\theta = \arccos x$

$$a_k = \frac{(f(x), T_k(x))}{(T_k(x), T_k(x))}$$

Now on $[0,1]$:

shift Chebyshev polynomials to interval $[0,1]$

$s \in [0,1] \quad \leftrightarrow \quad x = 2s - 1 \quad \in [-1, 1]$

$\Rightarrow$ consider $T_k(2s-1)$ voor $s \in [0,1]$

Chebyshev approximation:

$$C_n(s) = \sum_{k=0}^{n} a_k T_k(2s-1) \qquad s \in (0,1)$$

$$a_k = \frac{(f(s), T_k(2s-1))}{(T_k(2s-1), T_k(2s-1))}$$

inner product $(f,g) = \int_0^1 \frac{1}{\sqrt{1-(2s-1)^2}} f(s)\, g(s)\, ds$

replacing s with x results in required answer

c. $a_0 = 1 - \frac{1}{\pi}$ $\quad a_1 = \frac{1}{2}$

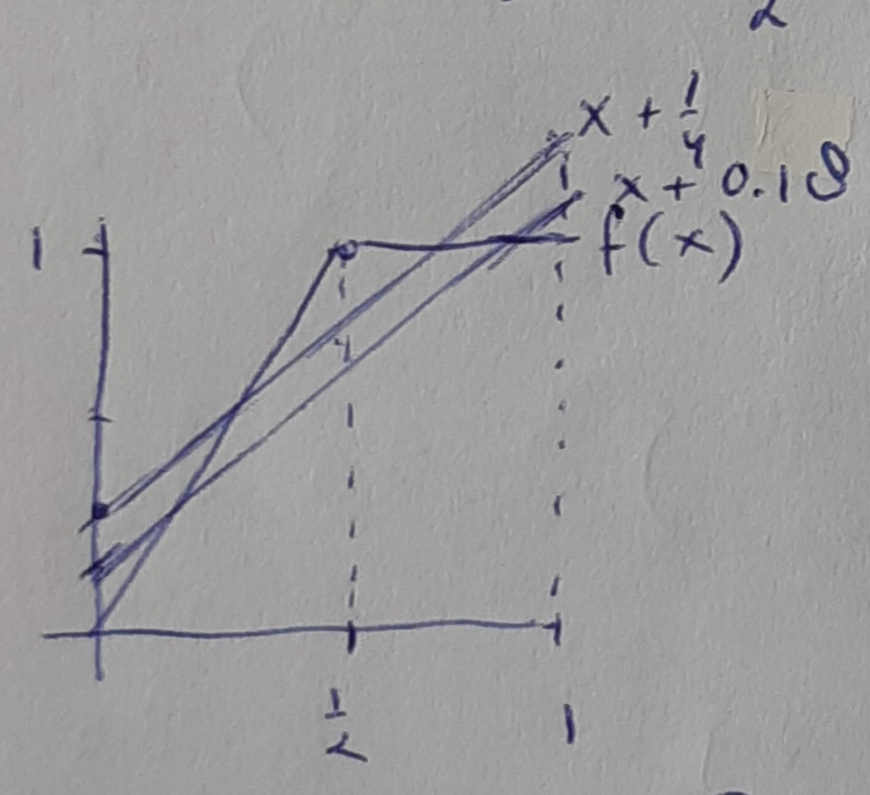
- $C_1(x) = \left(1 - \frac{1}{\pi}\right) T_0(2x-1) + \frac{1}{2} T_1(2x-1)$

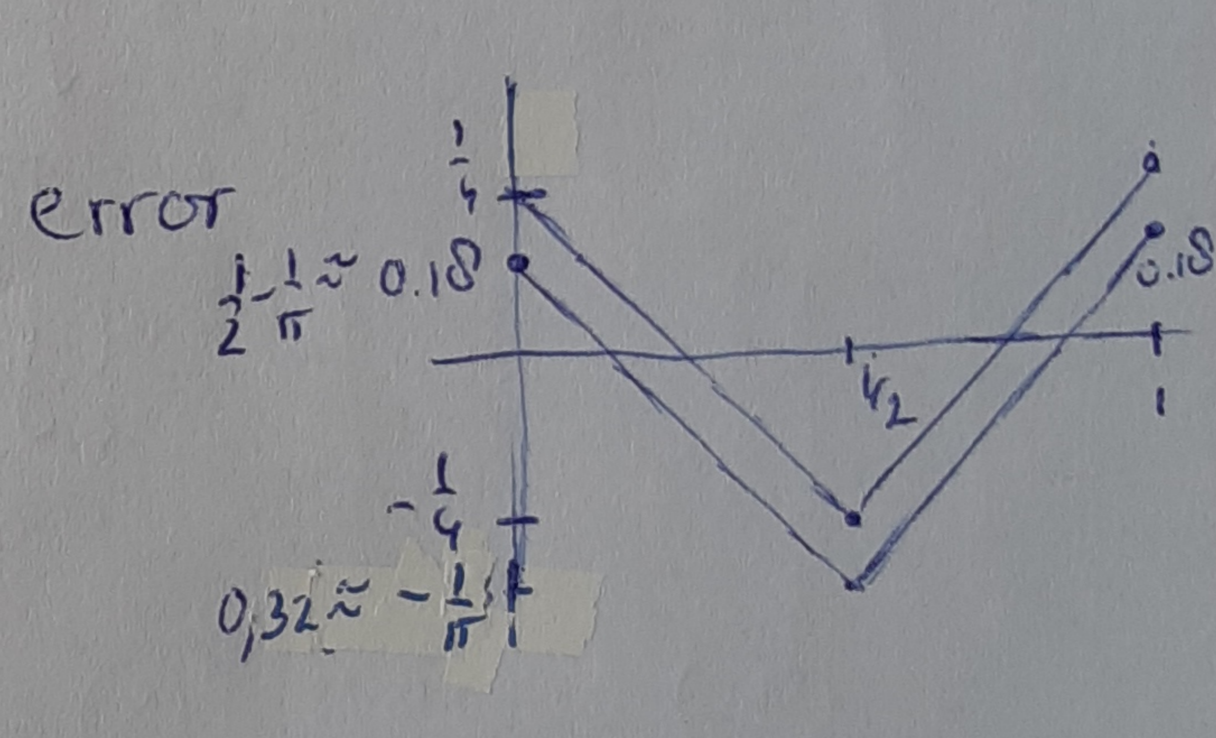
  note: $T_0(x) = 1 \Rightarrow T_0(2x-1) = 1$

  $T_1(x) = x \Rightarrow T_1(2x-1) = 2x-1$

$$\Rightarrow C_1(x) = \left(1 - \frac{1}{\pi}\right) + \frac{1}{2}(2x-1) = \frac{1}{2} - \frac{1}{\pi} + x \approx 0.18 + x$$

- plot:





- theorem of la Vallée - Poussin

  let $n \geq 0$ let $x_0 < x_1 < \dots < x_{n+1}$ be $n+2$ points on $[a,b]$

  if there exists a polynomial q_n of degree $\leq n$ s.t

  $$f(x_j) - q_n(x_j) = (-1)^j e_j \qquad j = 0, 1, \dots, n+1$$

  where all e_j have the same sign and are non null, then

  $$\min_{0 \leq j \leq n+1} |e_j| \leq E_n^*(f)$$

  where $E_n^*(f) = \| f - p_n^* \|_\infty$ with p_n^* polynomial of best approximation for $f \in C^0([a,b])$

now $n=1$, error oscillating, lower bound for $E_n^*(f)$ is minimum of extrema $\Rightarrow \frac{1}{2} - \frac{1}{\pi} \approx 0.18 \leq E_n^*(f)$

p_n^*, $E_n^*(f)$ best approximation $\Rightarrow E_n^*(f) \leq \max |e_j| = |f(1/2 + 1/2 - \frac{1}{\pi}) - 1| = \frac{1}{\pi} \approx 0,32$

(d) the first derivative of $f(x)$ is not continuous
$\Rightarrow f \notin C^\infty$
$\Rightarrow$ the convergence is not faster than any positive power of n

(e) i. $$(f, T_k(-1+2x)) = \int_0^1 \frac{1}{\sqrt{1-(-1+2x)^2}} f(x) T_k(-1+2x)\, dx$$

$T_k(x) = \cos k\theta \quad \theta = \arccos x$

$$\Rightarrow (f, T_k(-1+2x)) = \int_0^1 \frac{1}{\sqrt{1-(-1+2x)^2}} f(x) T_k(-1+2x)\, dx$$

substitue $s = -1+2x \quad \Rightarrow x = \frac{1+s}{2} \quad dx = \frac{1}{2} ds$

$s: -1 \to 1$

$$\Rightarrow (f, T_k(-1+2x)) = \int_{-1}^1 \frac{1}{\sqrt{1-s^2}} f\left(\frac{1+s}{2}\right) T_k(s) \frac{1}{2} ds$$

$T_k(s) = \cos k\theta, \quad \theta = \arccos s \Rightarrow s = \cos\theta$

substitute $s = \cos\theta \quad ds = -\sin\theta\, d\theta$

$s: \pi \to 0$

$$= \int_\pi^0 \frac{1}{\sqrt{1-\cos^2\theta}} f\left(\frac{1+\cos\theta}{2}\right) T_k(\cos\theta) \cdot \frac{1}{2} \cdot -\sin\theta\, d\theta$$

$\frac{1}{\sqrt{\sin^2\theta}} = \frac{1}{\sin\theta}$ on $[0,\pi]$ $\qquad \cos k \arccos \cos\theta = \cos k\theta$

$$= \frac{1}{2}\int_0^\pi f\left(\frac{1+\cos\theta}{2}\right) \cos k\theta\, d\theta$$

(ii) if $f(x) = T_k(-1+2x)$ in formula of i. replace $f\left(\frac{1+\cos\theta}{2}\right)$ by $\cos k\theta$

$$\Rightarrow (T_k(-1+2x), T_k(-1+2x)) = \frac{1}{2}\int_0^\pi (\cos k\theta)^2\, d\theta$$

(iii) $$a_0 = \frac{(f(x), T_0(-1+2x))}{(T_0(-1+2x), T_0(-1+2x))}$$

$\theta: 0 \to \frac{1}{2}\pi \qquad \frac{1+\cos\theta}{2}: 1 \to \frac{1}{2}$

$\frac{1}{2}\pi \to \pi \qquad \frac{1+\cos\theta}{2}: \frac{1}{2} \to 0$

$$(f(x), T_0(-1+2x)) = \frac{1}{2}\int_0^\pi f\left(\frac{1+\cos\theta}{2}\right) \cos 0\theta\, d\theta = \frac{1}{2}\int_0^\pi f\left(\frac{1+\cos\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}\pi} 1\, d\theta + \frac{1}{2}\int_{\frac{1}{2}\pi}^\pi 2\left(\frac{1+\cos\theta}{2}\right) d\theta = \frac{1}{4}\pi + \frac{1}{2}(\theta + \sin\theta\Big|_{\frac{1}{2}\pi}^\pi$$

$$= \frac{1}{4}\pi + \frac{1}{2}\pi + \frac{1}{2}\sin\pi - \frac{1}{2}\cdot\frac{1}{2}\pi - \frac{1}{2}\sin\frac{1}{2}\pi = \frac{1}{2}\pi - \frac{1}{2}$$

$$(T_0(-1+2x), T_0(-1+2x)) = \frac{1}{2}\int_0^\pi (\cos 0\theta)^2 d\theta = \frac{1}{2}\int_0^\pi d\theta = \frac{1}{2}\pi$$

$$a_0 = \frac{\frac{1}{2}\pi - \frac{1}{2}}{\frac{1}{2}\pi} = 1 - \frac{1}{\pi}$$